Poisson Disk Sampling on the Grassmannian: Applications in Subspace Optimization

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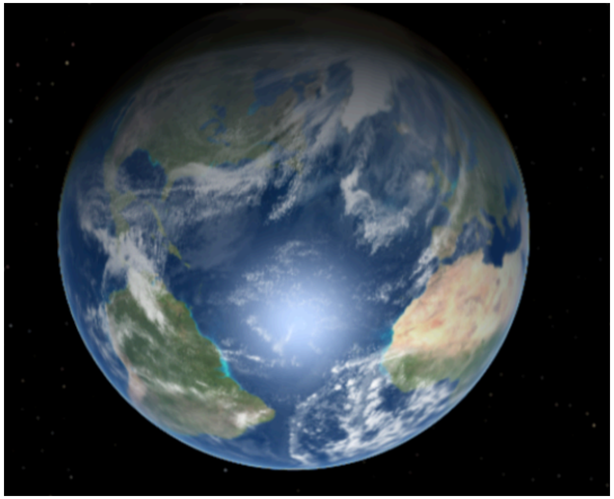
Non-Euclidean Manifolds in Computer Vision

- Why should we care? Because, we cannot ignore the real geometry of the mathematical spaces.



What is the distance between Paris and San Francisco?

- An N-dimensional **manifold** M is a topological space where every point is endowed with “local” Euclidean structure

Optimization on Non-Linear Manifolds

- *Matrix Lie Group:* Set of *n x n* non-singular matrices with a smooth manifold structure

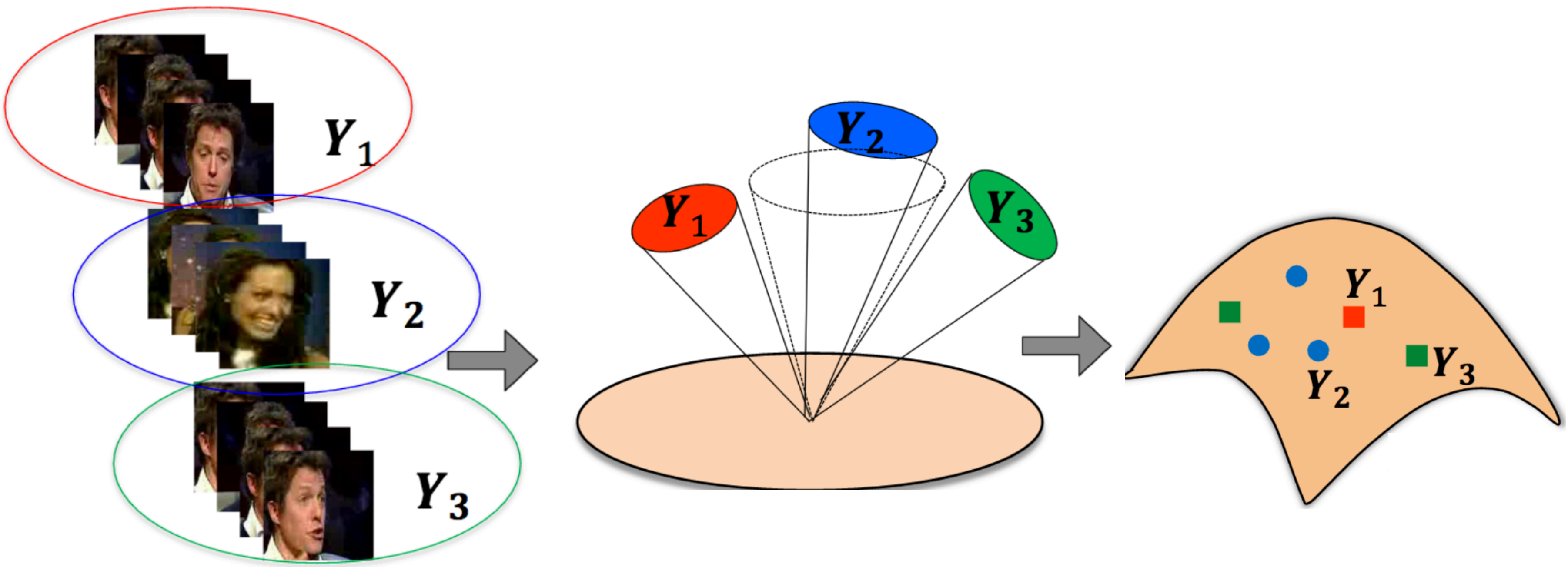
$$\mathcal{SO}(n) = \left\{ Y \in \mathbb{R}^{n \times n} : Y^T Y = I,\ det(Y) = 1 \right\}$$

A truck rendered at different orientations on *SO(2)* – Rotation Matrices

- *Stiefel Manifold:* Set of orthonormal matrices of size *n x p*

- *Grassmann Manifold:* Set of p-dimensional linear subspaces in *n*-dimensions

Subspace Analysis on Grassmann Manifolds



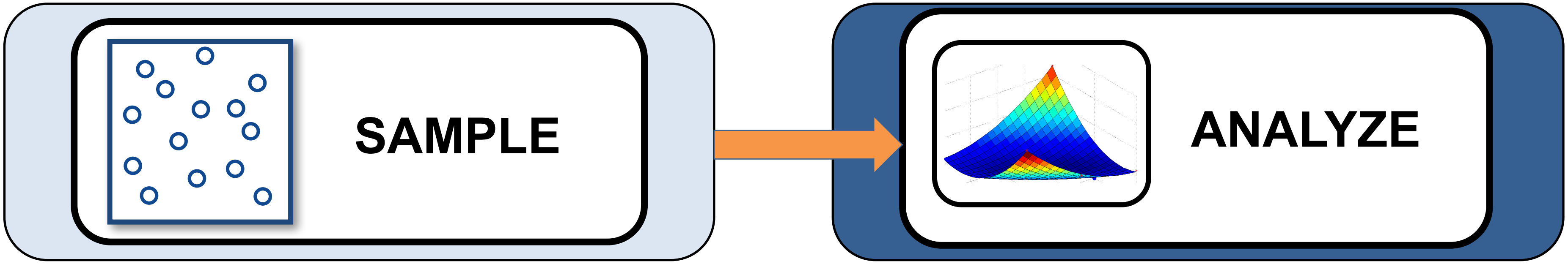
Videos can be treated as image sets

Each set modeled as a linear subspace

Analysis on the Grassmann manifold

Gradient descent based optimization requires understanding of the characteristics of loss functions on non-Euclidean manifolds

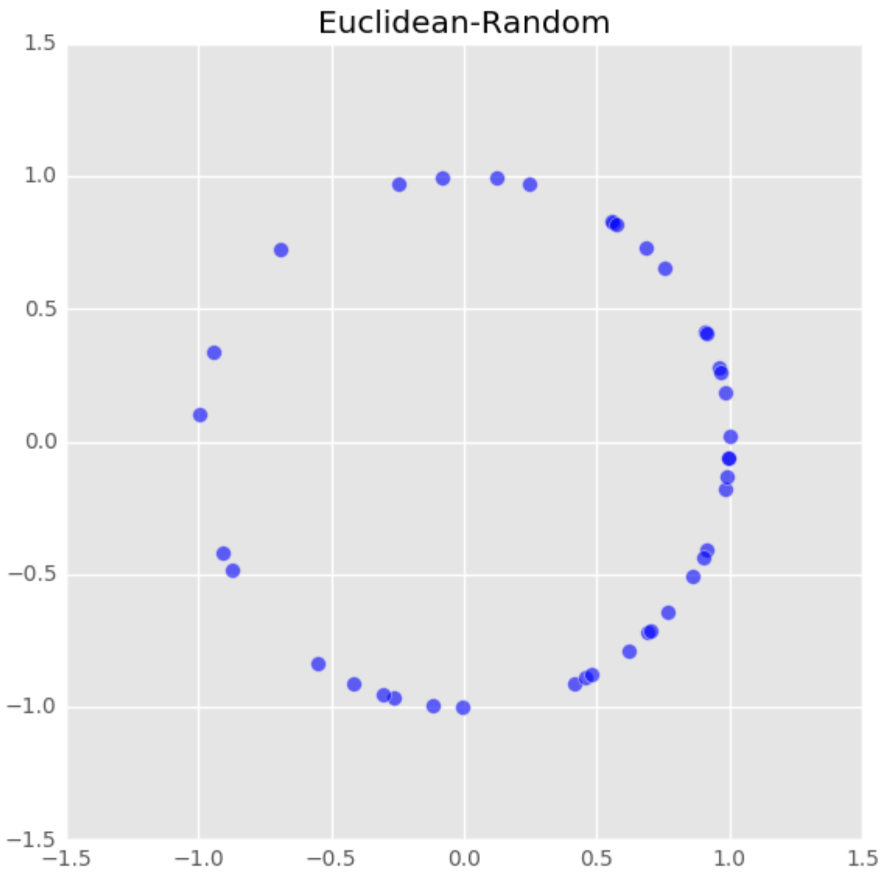
A Statistical Approach

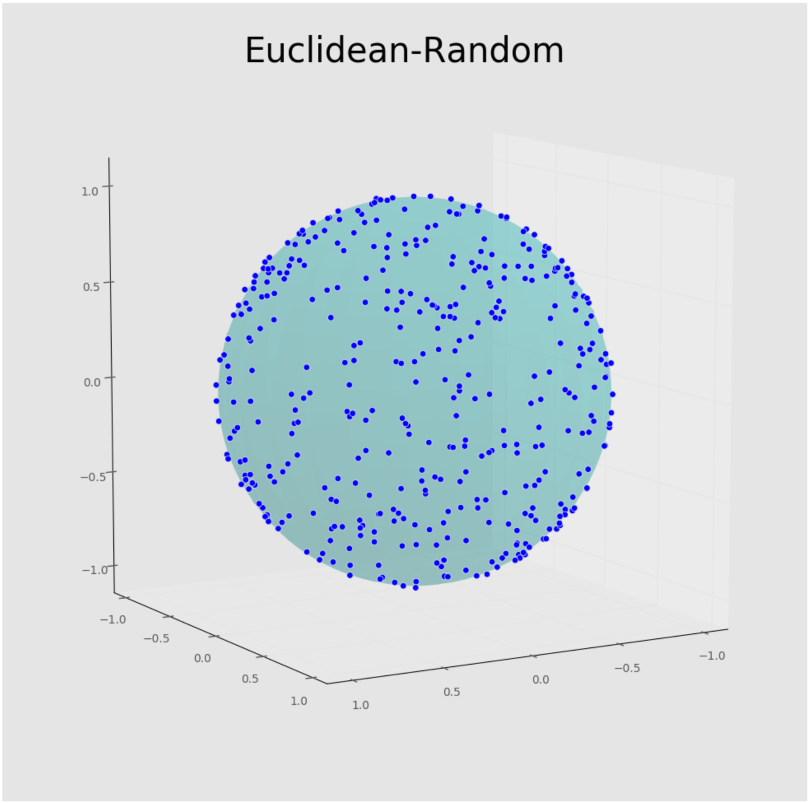


Improved Sample Designs Can Enable a Better Understanding of the Optimization Surface

Existing Techniques Provides Poor Coverage

Projecting random samples created in the Euclidean space results in poor coverage of the manifold





Characteristics of a Good Sampling Pattern

- In its most generic form the inference problem can be described as recovering or analyzing a multi-variate, smooth function

$$f : \mathcal{M} \mapsto \mathbb{R}^d$$

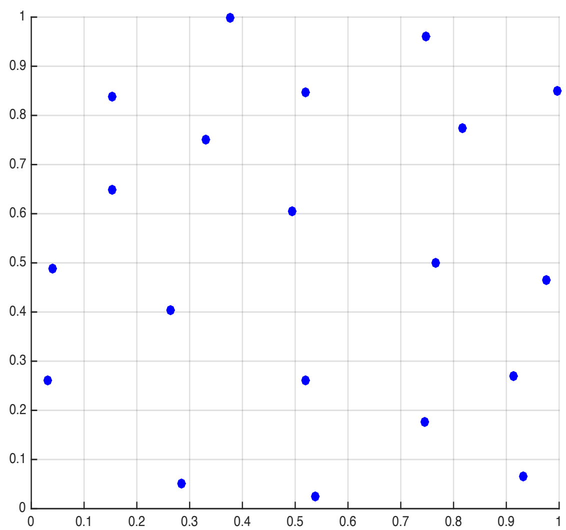
defined on some manifold $\mathcal{M} \subset \mathbb{R}^n$.

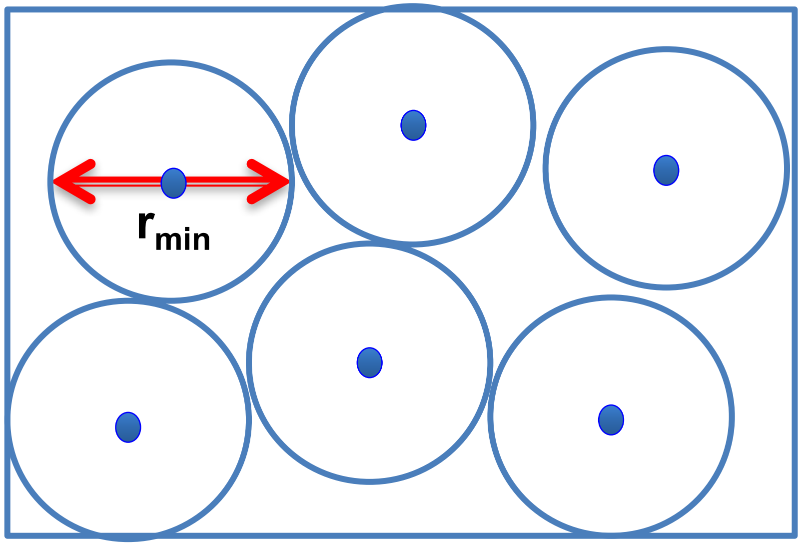
- A good sampling pattern is expected to have two main properties
  1. the sampling should be random
  2. the samples should uniformly cover all the manifold

Sampling Euclidean Spaces: Poisson Disk Sampling

Generating Poisson Disk Samples

- Each sample is placed with uniform probability
- No two samples are closer than a pre-specified radius
- A PDS is maximal when no more points can be inserted





Grassmann Manifolds

- Linear subspaces of dimension *k* in *n*-dimensional Euclidean spaces are points on the Grassmannian *Gr*(*n*,*k*)

Intrinsic Distance on the Grassmannian

Given two subspaces $\mathbf{A}, \mathbf{B} \in Gr(n,k)$, the geodesic distance between them is measured as $d(\mathbf{A}, \mathbf{B}) = (\sum_{i=1}^{k} \theta_i^2)^{1/2}$, where $\{\theta_i\}_{i=1}^{k}$ are the principal angles.

- Measures the smallest rotation that takes from one subspace to another
- Chordal distance: $(\sum_{i=1}^{k} \sin^2 \theta_i)^{1/2} = \frac{1}{\sqrt{2}} \|\mathbf{A}\mathbf{A}^T - \mathbf{B}\mathbf{B}^T\|_F.$

Dart throwing for Poisson Disk Sampling

Algorithm 1 Dart Throwing on the Grassmann manifold

Require: Dimensions (n, k), number of samples N and r_{min}, $\mathbf{S} = \emptyset$

1: **while** $|\mathbf{S}| \leq N$ **do**

2: Throw a Dart i:

- Generate random matrix $\mathbf{Z}_i \in \mathbb{R}^{n \times k}$
- Obtain the corresponding point $Q \in \mathcal{G}_{n,k}$ as the QR decomposition of $\mathbf{Z}_i$.
- Assign $\mathbf{S}_i \leftarrow Q$

3: **if** $d_{\mathcal{G}}(\mathbf{S}_i, \mathbf{S}_j) \leq r_{min}$, $\forall \mathbf{S}_j \in \mathbf{S}$ **then**

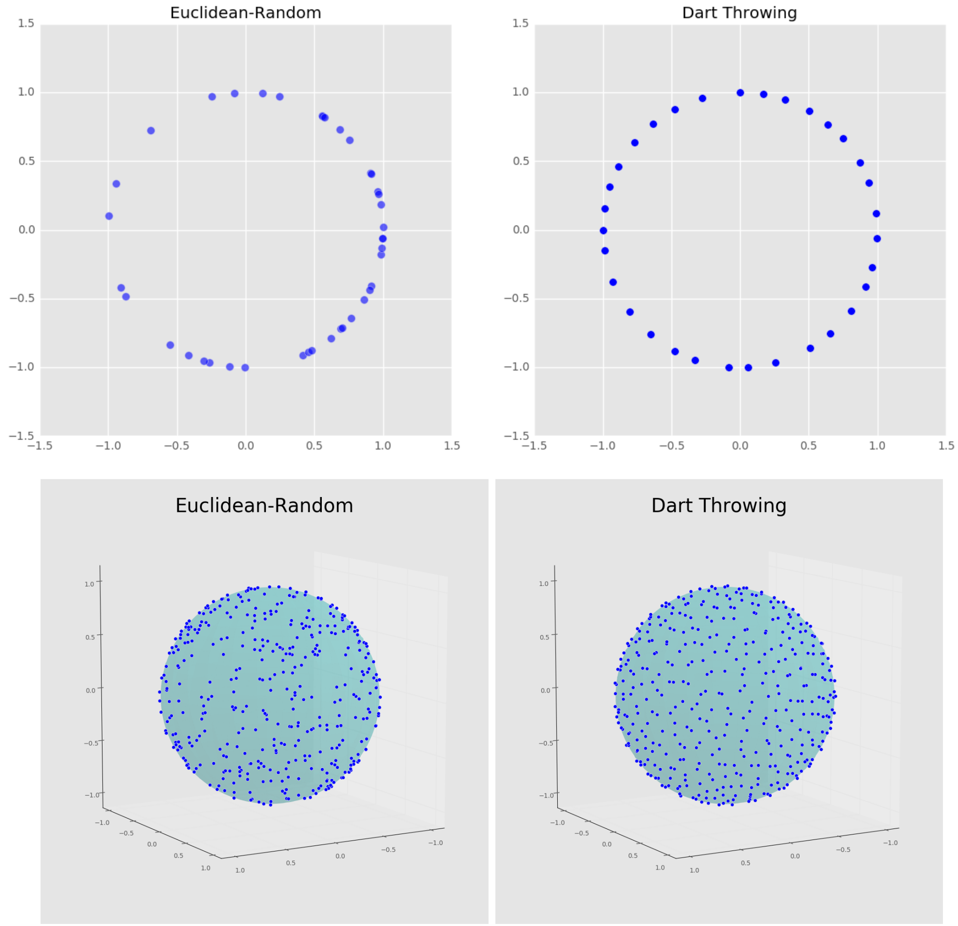
4: Add sample $\mathbf{S}_i$ to the point set $\mathbf{S}$

5: **end if**

6: **end while**

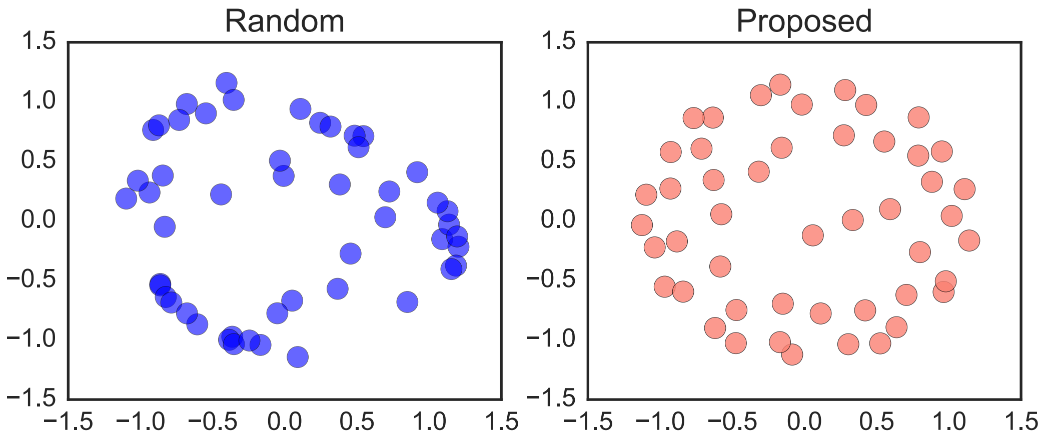
Proposed Scheme Provides Better Coverage

- Our proposed approach produces better uniform samples, thereby leading to improved analysis and optimization

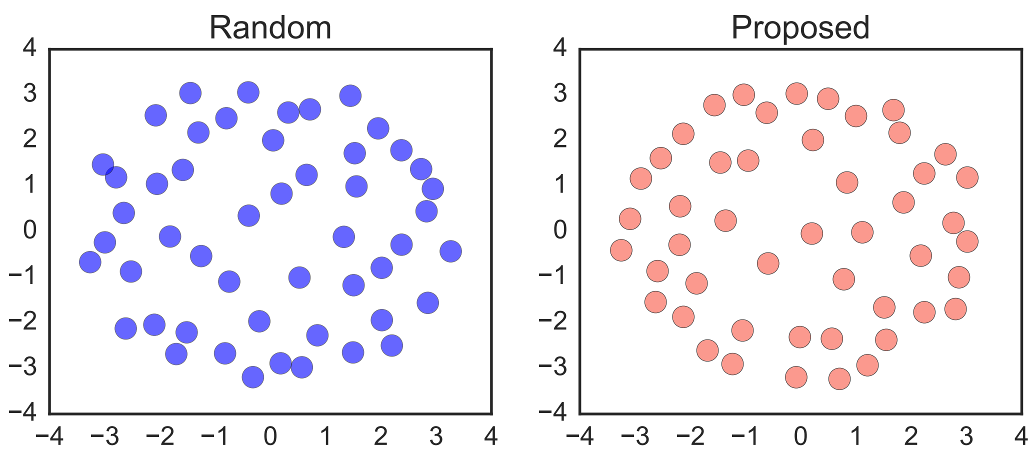


Visualizing the sample distributions (MDS)

50 samples from G_3^2



50 samples from G_8^4



Subspace Optimization using Grassmann Samples

- Subspace Learning as Graph Embedding

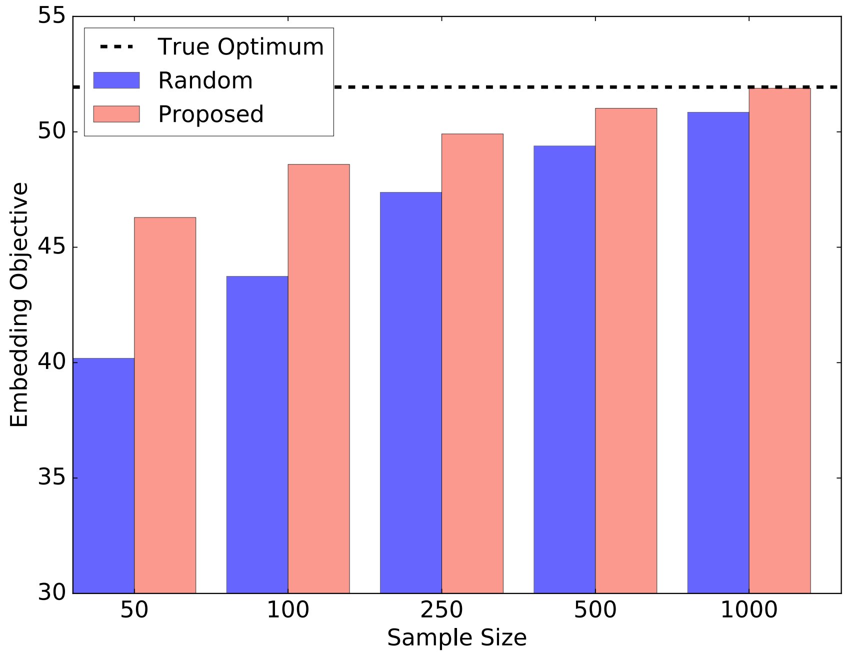
$$\mathbf{Y}^* = arg \min_{tr(\mathbf{V}^T\mathbf{X}\mathbf{B}\mathbf{X}^T\mathbf{V})=d} tr(\mathbf{V}^T\mathbf{X}\mathbf{L}\mathbf{X}^T\mathbf{V})$$

- Solution of PCA/LDA is approximated as a linear combination of the Dart throwing samples $\mathbf{S}_\mathbf{i}$
- Consensus based approach

$$\mathbf{V}^* = arg \min_{\mathbf{V}^T\mathbf{V}=\mathbf{I}} tr\left(\mathbf{V}^T\sum_{i=1}^{N}\left(\alpha_i\mathbf{I} - \alpha_i\mathbf{S}_i\mathbf{S}_i^T\right)\mathbf{V}\right)$$

PDS Samples are Very Effective in Preserving the Desired Relationships

- Principal Component Analysis cost optimization using consensus on the Grassmannian (higher is better)



Ecoli Dataset (7 dimensional)

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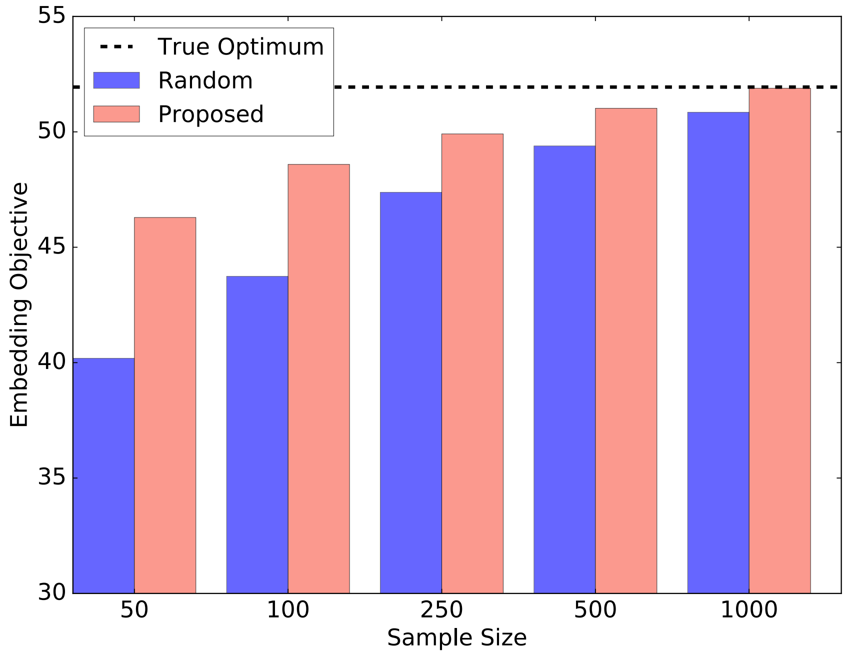

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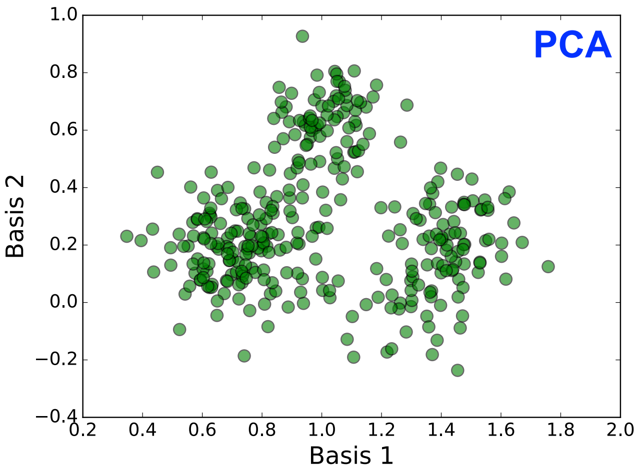
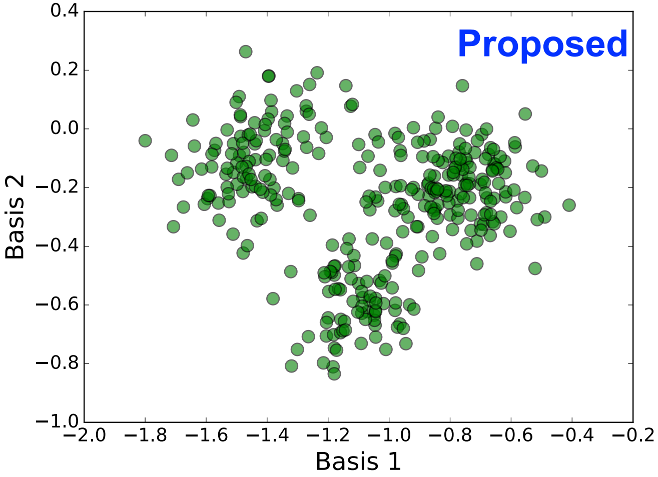
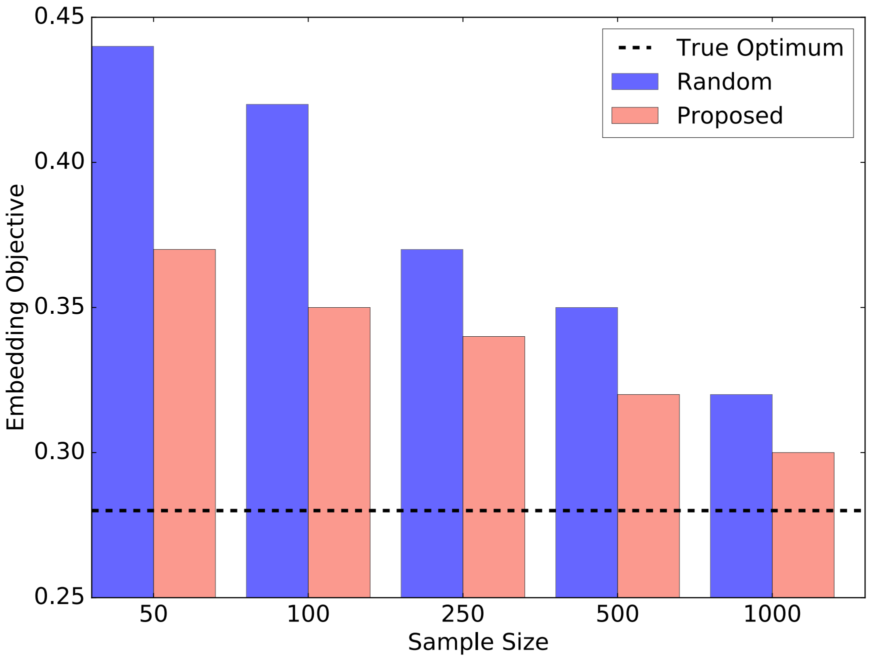




PDS Samples are Very Effective in Preserving the Desired Relationships

- Comparison of the consensus embedding obtained using the LDA cost on the Grassmannian, to the true solution obtained using generalized eigenvalue decomposition



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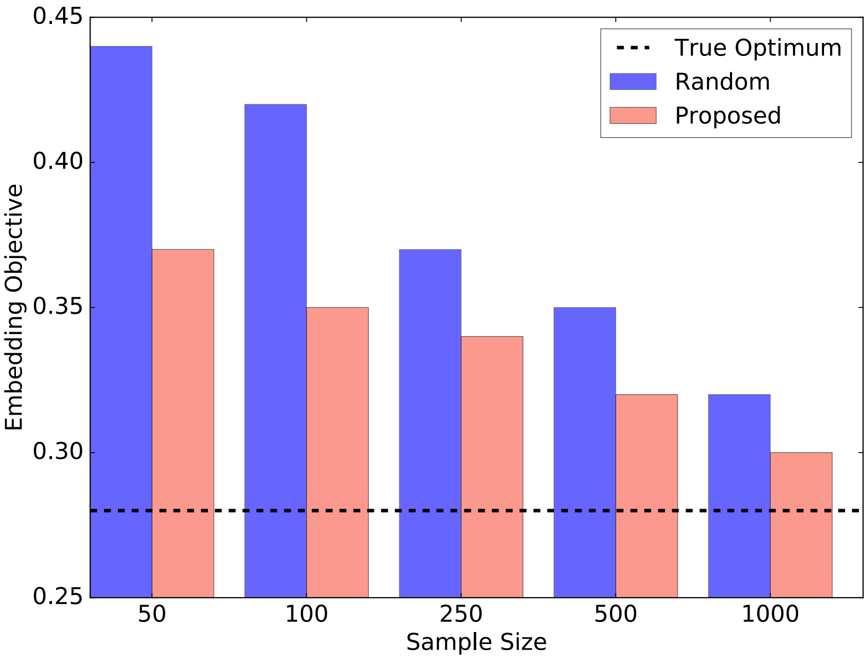

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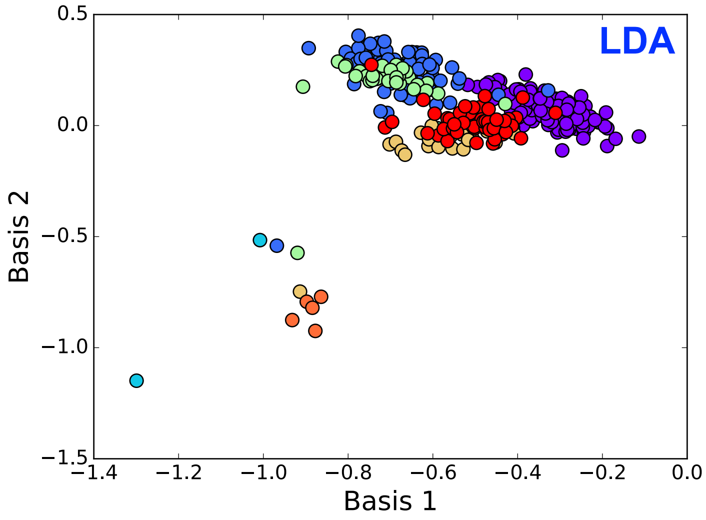
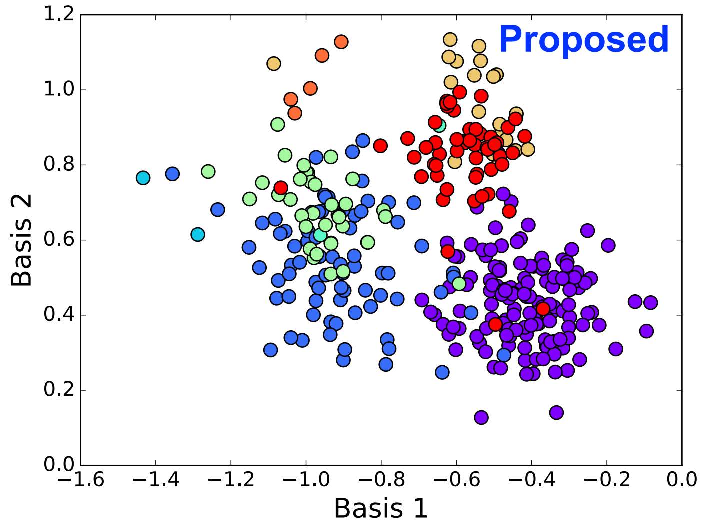
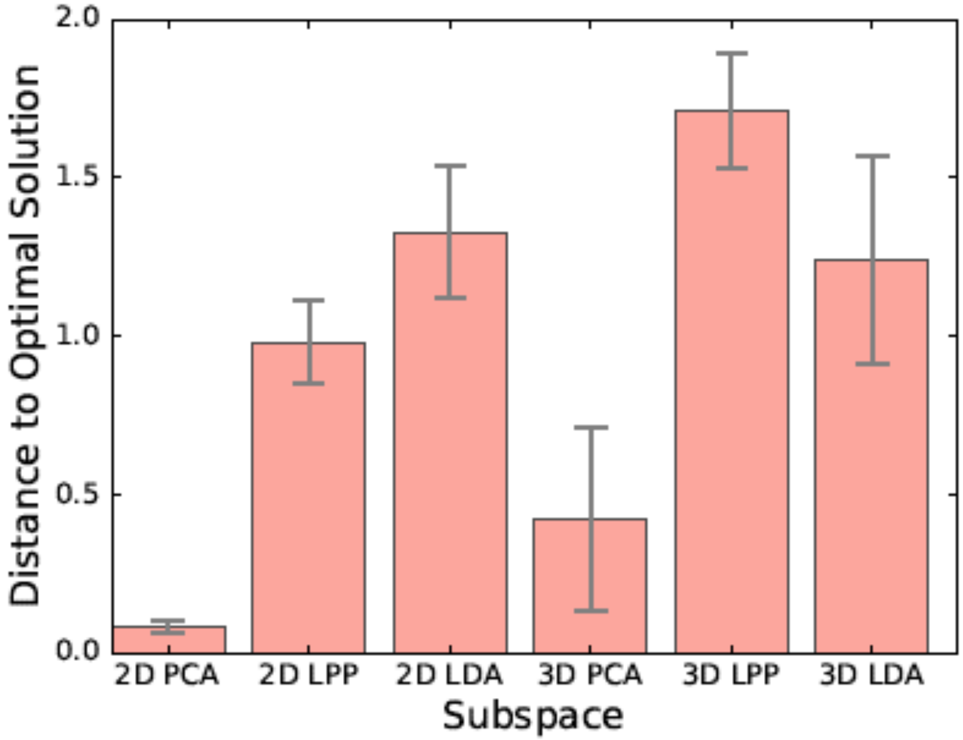




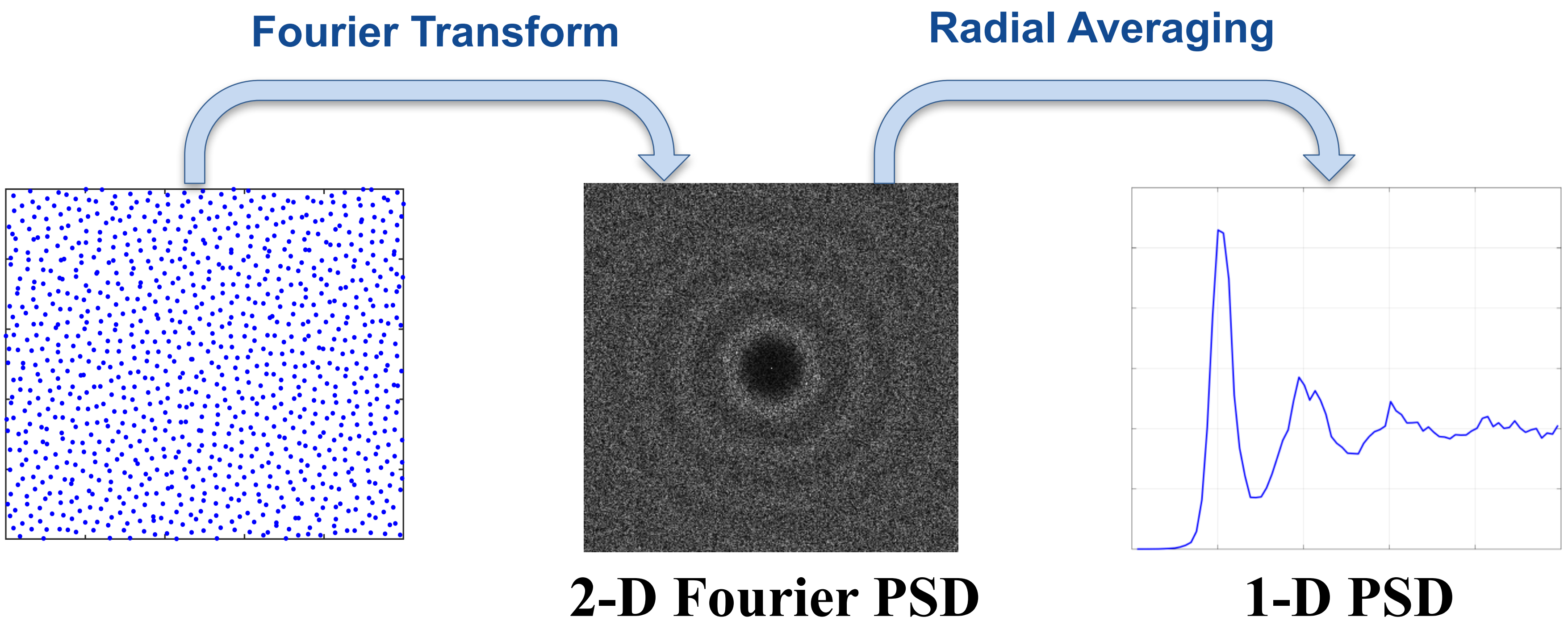
Diverse Solutions

- Grassmann distance between the optimal solution and the linear subspace inferred using the consensus from PDS samples for the Ecoli dataset.



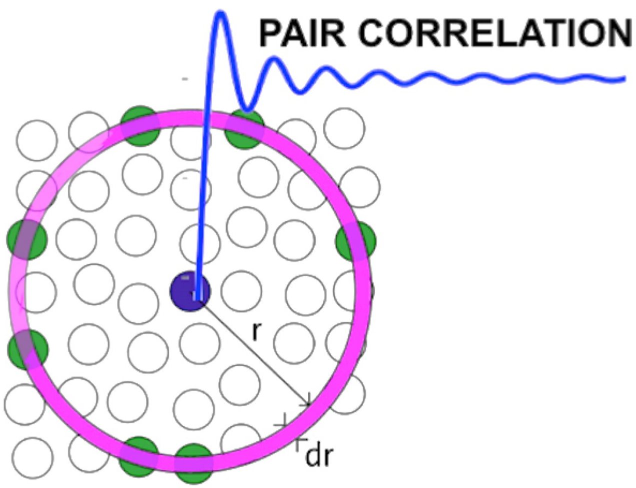
Why Does PDS Work?

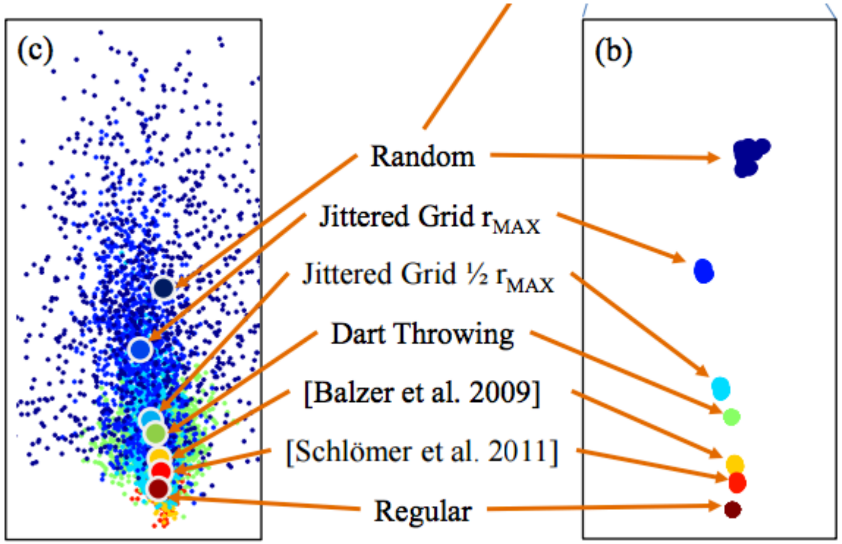
- Sample design can be directly viewed from a ***function reconstruction*** perspective – Fourier Domain Analysis



Characterizing Spatial Statistics

- In statistical mechanics, Pair Correlation Function describes how density varies as a function of distance from a reference

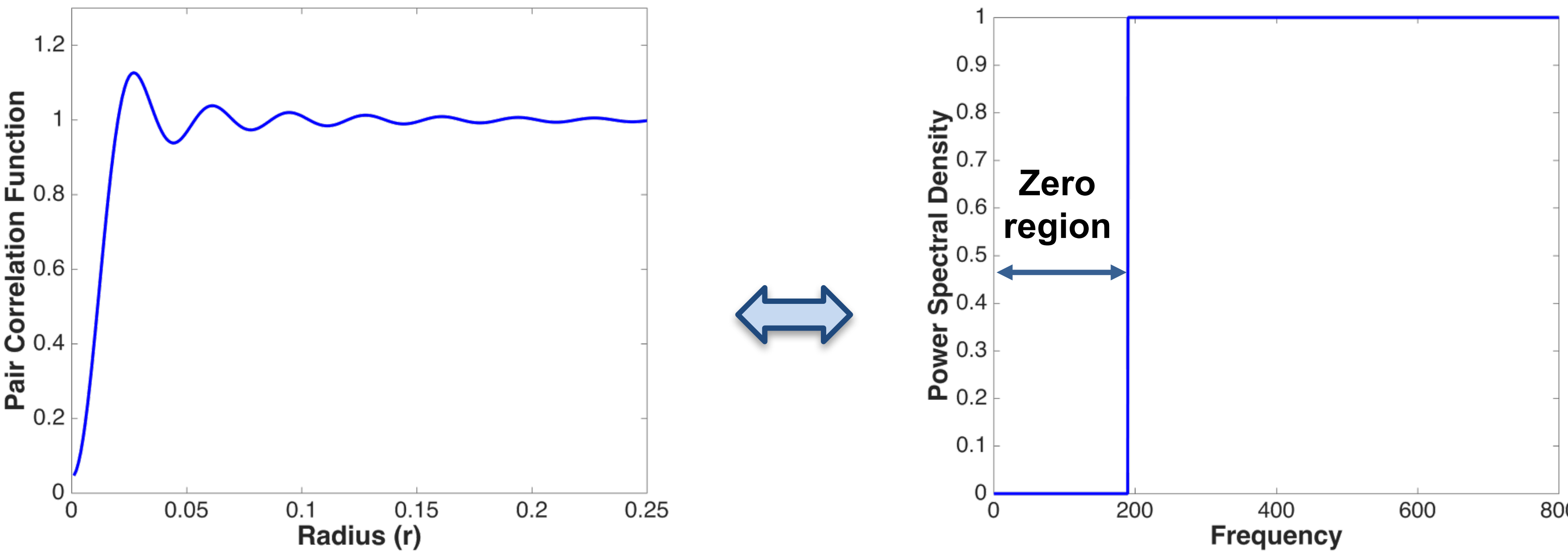




Key Idea for Spectral Sampling

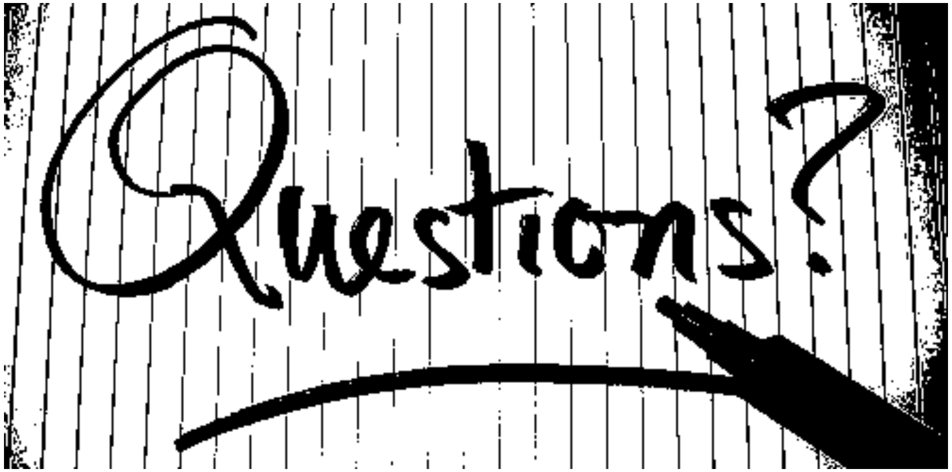
- 1-D PSD and PCF are connected via the *1-D* Hankel transform

$$P(k) = 1 + \rho(2\pi)^{\frac{d}{2}} k^{1-\frac{d}{2}} H_{\frac{d}{2}-1}\left(r^{\frac{d}{2}-1}(G(r)-1)\right)$$



Summary

- Poisson disk sampling for better coverage of Grassmann
- Effective approximation of the optimal solution for subspace learning problems
- High-quality samples can serve as anchor points to search through the Grassmannian using conventional optimization strategies



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